OPTIMAL FORM OF SHIELDING FOR POLYCHROMATIC γ-RADIATION

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In [1] the problem of the construction of the optimal form of shielding (in the sense of minimum weight) for the monoenergetic γ-radiation of linear, disc and cylindrical sources was considered. Below, for the same source geometries, the problem of optimal form of shielding is solved for the case of polychromatic radiation. As in [1], radiation scattering in the environment is neglected, and the sources are assumed to radiate isotropically. Multiple scattering in the shielding is taken into account by using an analytical expression for the build-up factor.

1. Consider first a line source. The energy flux of γ-quanta at a point 0 (see Fig. 1) distant h from the line source of length $2r_0$ and specific intensity S is [2]

$$K = \frac{S}{2\pi h}\int_0^{\varphi_0}\sum_{i=1}^{j} n_i E_i B_i \exp(-\mu_i x)\,d\varphi. \tag{1.1}$$

Here x is the shield thickness and B_i is the energy build-up factor, taking account of multiple γ-ray scattering in the shield. We use an analytical expression for the energy build-up factor in the form [3]

$$B = A_1\exp(-\alpha_1\mu x) + A_2\exp(-\alpha_2\mu x). \tag{1.2}$$

Here A_1, A_2, α_1, α_2, are numerical coefficients. Substituting in (1.1) and putting $\mu_i' = \mu_i(1+\alpha_{1i})$ and $\mu_i'' = \mu_i(+\alpha_{2i})$,

$$K = \frac{S}{2\pi h}\int_0^{\varphi_0}\sum_{i=1}^{j} n_i E_i [A_{1i}\exp(-\mu_i' x) + A_{2i}\exp(-\mu_i'' x)]\,d\varphi. \tag{1.3}$$

Here j is the number of lines in the γ-spectrum, n_i is the yield of γ-quanta of given energy E_i per nuclear decay, and μ_i is the γ-ray attenuation coefficient in the material. The weight of shielding, assuming $x \ll R$, is

$$G = 2ht\rho\int_0^{\varphi_0} x\sec\varphi\,d\varphi, \tag{1.4}$$

where h is the distance from the shielded point 0 to the center of the source, ρ is the density of the shielding material, and t is the longitudinal dimension of the shield.

The problem is to determine the form of shielding which provides a given energy flux at the shielded point, $K = K_0$, with minimal mass G. We transform to dimensionless variables

$$\varkappa = \frac{2\pi K h}{SE},\quad g = \frac{G\mu}{2ht\rho},\quad \zeta = x\mu,\quad \lambda_i' = \frac{\mu_i'}{\mu},\quad \lambda_i'' = \frac{\mu_i''}{\mu},\quad \nu_i = \frac{E_i}{E}. \tag{1.5}$$

The absorption coefficient μ corresponds to the energy

$$E = \frac{n_1E_1 + \ldots + n_jE_j}{n_1 + \ldots + n_j}. \tag{1.6}$$

The integrals (1.3) and (1.4) may be expressed in the variables (1.5) as

$$\frac{d\varkappa}{d\varphi} = \sum_{i=1}^{j} n_i\nu_i [A_{1i}\exp(-\lambda_i'\zeta) + A_{2i}\exp(-\lambda_i''\zeta)],\quad \frac{dg}{d\varphi} = \zeta\sec\varphi, \tag{1.7}$$

with boundary conditions

$$\varkappa(0) = 0,\quad \varkappa(\varphi_0) = \varkappa_0,\qquad g(0) = 0 \quad (0 < \varphi_0 < {}^1\!/_2\pi). \tag{1.8}$$

We formulate the variational problem in the following way (cf. [1]): for the system (1.7) in a class of piecewise-continuous functions there is found a control parameter $\zeta(\varphi) \ge 0$, which gives a minimum finite value of the coordinate $g(\varphi_0)$ and satisfies the boundary conditions (1.8). The formulated variational problem will be solved using the method of L. S. Pontryagin [4]. We write the Hamiltonian

$$H = p_\varkappa\sum_{i=1}^{j} n_i\nu_i [A_{1i}\exp(-\lambda_i'\zeta) + A_{2i}\exp(-\lambda_i''\zeta)] + p_g\zeta\sec\varphi, \tag{1.9}$$

where the momenta $p_\varkappa$ and p_g are constants since

$$\frac{dp_\varkappa}{d\varphi} = -\frac{\partial H}{\partial\varkappa} = 0,\quad \frac{dp_g}{d\varphi} = -\frac{dH}{\partial g} = 0. \tag{1.10}$$

According to [4], the momentum p_g, corresponding to the minimized coordinate, may be put equal to -1 and $p_\varkappa$ to some constant p_0. Then

$$H = p_0\sum_{i=1}^{j}\psi_i(\zeta) - \zeta\sec\varphi,\quad \psi_i(\zeta) = n_i\nu_i[A_{1i}\exp(-\lambda_i'\zeta) + A_{2i}\exp(-\lambda_i''\zeta)] \tag{1.11}$$

On the basis of the maximum principle, the optimal control $\zeta(\varphi) \ge 0$ must give the absolute maximum of the function H [see (1.11)]. From analysis of the equation for the partial derivative

$$\frac{\partial H}{\partial\zeta} = -\left(p_0\sum_{i=1}^{j}\eta_i(\zeta) + \sec\varphi\right),$$

$$\eta_i(\zeta) = n_i\nu_i[\lambda_i' A_{1i}\exp(-\lambda_i'\zeta) + \lambda_i'' A_{2i}\exp(-\lambda_i''\zeta)],$$

it follows that on those intervals where

$$p_0[\eta_1(0) + \ldots + \eta_j(0)] \geqslant -\sec\varphi,$$

the maximum of H is reached for $\zeta = 0$. If

$$p_0[\eta_1(0) + \ldots + \eta_j(0)] \leqslant -\sec\varphi,$$

then $\zeta(\varphi)$ is determined from the condition $\partial H/\partial\zeta = 0$,

$$p_0[\eta_1(\zeta) + \ldots + \eta_j(\zeta)] + \sec\varphi = 0. \tag{1.12}$$

We denote the solution of (1.12) by $\zeta_0'(\varphi)$. The optimal form of shielding is then determined by the equations

$$\zeta(\varphi)=\begin{cases}\zeta_0'(\varphi) & \text{for } \cos\varphi\geqslant -1/p_0' \\ 0 & \text{for } \cos\varphi\leqslant -1/p_0'\end{cases}\quad (p_0'<0),$$

$$p_0'=p_0\sum_{i=1}^{j}\eta_i(0)=p_0\sum_{i=1}^{j}n_i\nu_i(\lambda_i'A_{1i}+\lambda_i''A_{2i}). \quad (1.13)$$

If $\cos\varphi_0\leqslant -1/p_0'$, then the optimal control $\zeta(\varphi)$ includes both parts of (1.13). The values of p_0' (or p_0) are determined from (1.12) with the condition $\zeta(\varphi_*) = 0$

$$p_0'=-\sec\varphi_*, \text{ for } p_0=-\frac{\sec\varphi_*}{\eta_1(0)+\ldots+\eta_j(0)}. \quad (1.14)$$

Substituting in (1.12), we get

$$\sum_{i=1}^{j}\eta_i(\zeta)=\frac{\cos\varphi_*}{\cos\varphi}\sum_{i=1}^{j}\eta_i(0), \quad (1.15)$$

and the optimal control $\zeta(\varphi)$ will be determined by

$$\zeta(\varphi)=\begin{cases}\zeta_0(\varphi) & \text{for } 0\leqslant\varphi\leqslant\varphi_* \\ 0 & \text{for } \varphi_*\leqslant\varphi\leqslant\varphi_0\end{cases}. \quad (1.16)$$

Here $\zeta_0(\varphi)$ is the solution of (1.15).

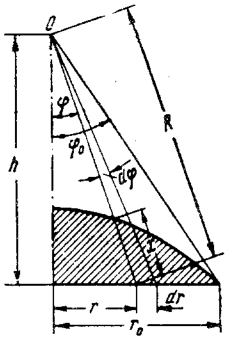

Fig. 1

Equation (1.16) is valid within the γ-quantum energy flux range

$$\int_0^{\varphi_0}\sum_{i=1}^{j}\psi_i(\zeta)\,d\varphi\leqslant\varkappa_0\leqslant\varphi_0\sum_{i=1}^{j}n_i\nu_i. \quad (1.17)$$

If $\varkappa_0$ is less than (1.17), i.e.,

$$0<\varkappa_0\leqslant\int_0^{\varphi_0}\sum_{i=1}^{j}\psi_i(\zeta)\,d\varphi, \quad (1.18)$$

then the thickness of shielding is everywhere greater than zero ($\varphi_*\geqslant\varphi_0$). In this case p_0 is determined. We rewrite (1.12)

$$-p_0\cos\varphi\,[\eta_1(\zeta)+\ldots+\eta_j(\zeta)]=1. \quad (1.19)$$

Then, taking account of (1.19), the first solution of (1.7) may be written

$$\frac{d\varkappa}{d\varphi}=-p_0\cos\varphi\sum_{i=1}^{j}\eta_i(\zeta)\sum_{i=1}^{j}\psi_i(\zeta),$$

$$\varkappa_0=-p_0\int_0^{\varphi_0}\cos\varphi\sum_{i=1}^{j}\psi_i(\zeta)\sum_{i=1}^{j}\eta_i(\zeta)\,d\varphi. \quad (1.20)$$

We determine the constant p_0 from (1.20) and substitute it in (1.12)

$$\varkappa_0\cos\varphi\sum_{i=1}^{j}\eta_i(\zeta)=\int_0^{\varphi_0}\cos\varphi\sum_{i=1}^{j}\psi_i(\zeta)\sum_{i=1}^{j}\eta_i(\zeta)\,d\varphi. \quad (1.21)$$

Solving (1.21) with reference to $\zeta(\varphi)$, the optimal form of the shield is obtained. The weight of the shield is found by integrating the second equation of (1.7). For this purpose, $\zeta(\varphi)$ is substituted from (1.16) in the interval (1.17) and from (1.21) in the interval (1.18).

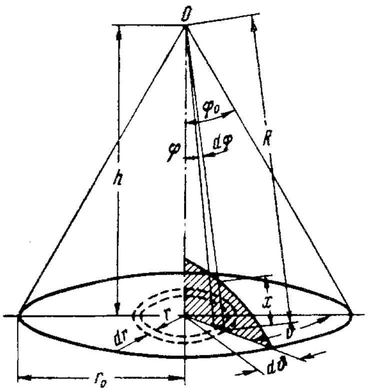

Fig. 2

2. Turning to the case of a disc source, the energy flux of γ-quanta at point 0 (see Fig. 2) is determined by [2]:

$$K=\frac{1}{2}\Gamma\int_0^{\varphi_0}\operatorname{tg}\varphi\sum_{i=1}^{j}\psi_i(\zeta)\,d\varphi, \quad (2.1)$$

[the form of function $\psi(\zeta)$ is as in (1.11)]. The weight of shielding G on the assumption $x \ll R$ is

$$G=2\pi h^2\rho\int_0^{\varphi_0}\operatorname{tg}\varphi\sec\varphi\,x\,d\varphi. \quad (2.2)$$

Here h is the distance from the shielded point O to the center of the source, ρ is the density of the shielding material, and Γ is the surface intensity of the disc source.

The dimensionless variables are as in (1.5) with the exception of $\varkappa$ and g, which are now

$$\varkappa=\frac{2K}{\Gamma E},\ g=\frac{G\mu}{2\pi h^2\rho}. \quad (2.3)$$

As in the case of a line source, the integrals (2.1) and (2.2) are expressed in the dimensionless variables (1.5) and (2.3) as the system

$$\frac{d\varkappa}{d\varphi}=\operatorname{tg}\varphi\sum_{i=1}^{j}\psi_i(\zeta),\qquad \frac{dg}{d\varphi}=\operatorname{tg}\varphi\sec\varphi\,\zeta. \quad (2.4)$$

The boundary conditions are as in (1.8).

According to the maximum principle, the optimal control, $\zeta(\varphi)\geq 0$, must give an absolute maximum of the function

$$H = \operatorname{tg}\varphi\,\{p_0\,[\psi_1(\zeta) + \ldots + \psi_j(\zeta)] - \zeta\sec\varphi\}. \quad (2.5)$$

Since the further argument is analogous to that given above for a line source, we shall restrict ourselves to the expression of the end results.

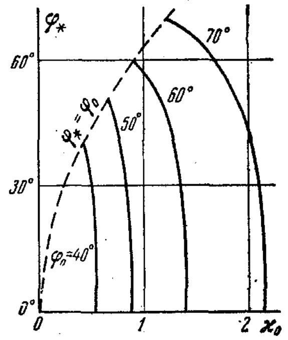

Fig. 3

The solution for a disc source agrees completely with (1.16). Integration of the first equation in (2.4) gives

$$\varkappa_0 = \int_0^{\varphi_*} \operatorname{tg}\varphi \sum_{i=1}^{j} \psi_i(\zeta)\,d\varphi + \left(\ln\frac{\cos\varphi_*}{\cos\varphi_0}\right)\sum_{i=1}^{j} n_i\nu_i. \quad (2.6)$$

The solution for a disc source (1.16) is valid over the range of flux

$$\int_0^{\varphi_0} \operatorname{tg}\varphi \sum_{i=1}^{j}\psi_i(\zeta)\,d\varphi \leqslant \varkappa_0 \leqslant (\ln\sec\varphi_0)\sum_{i=1}^{j} n_i\nu_i. \quad (2.7)$$

As an example, Fig. 3 shows a graph of the dependence of φ_* on $\varkappa_0$ for the Co^{60} γ-spectrum (with a lead shield). As in (1.21), we obtain

$$\varkappa_0\cos\varphi \sum_{i=1}^{j}\eta_i(\zeta) = \int_0^{\varphi_0}\sin\varphi\sum_{i=1}^{i}\psi_i(\zeta)\sum_{i=1}^{j}\eta_i(\zeta)\,d\varphi, \quad (2.8)$$

the solution of which $\zeta(\varphi)$ is valid in the interval

$$0 < \varkappa_0 \leqslant \int_0^{\varphi_0}\operatorname{tg}\varphi\sum_{i=1}^{j}\psi_i(\zeta)\,d\varphi. \quad (2.9)$$

The weight of shielding is determined in the same way as for a line source. Figure 4 shows a graph of $g_0(\varkappa_0)$ for fixed values of φ_0 for the Co^{60} γ-spectrum (with a lead shield). The broken curve, for $\varphi_* = \varphi_0$, separates regions (2.7) and (2.9).

3. We now find the optimal form of shielding for a cylindrical source. With self-absorption, the energy flux of γ-quanta at point O (see Fig. 5) from a cylindrical source is [5]

$$K = \frac{1}{2}\gamma\left\{\int_0^{\varphi_k}\sin\varphi\sum_{i=1}^{j}\frac{B_i n_i E_i}{\mu_{\varepsilon i}}\exp(-\mu_{\tau i}x)\,[1 - \right.$$

$$-\exp(-\mu_{\varepsilon i}b\sec\varphi)]\,d\varphi + \int_{\varphi_k}^{\varphi_0}\sin\varphi\sum_{i=1}^{j}\frac{B_i n_i E_i}{\mu_{\varepsilon i}}\times \quad (3.1)$$

$$\left.\times\exp(-\mu_{\tau i}x)\left[1 - \exp\left(-\frac{\mu_{\varepsilon i}r_0}{\sin\varphi}(1 - \operatorname{tg}\varphi\operatorname{ctg}\varphi_0)\right)\right]d\varphi\right\}.$$

(3.1) (cont'd.)

Here γ is the volume intensity of the source, and the indices ε and τ refer respectively to source and shield.

To take account of multiple scattering in the shield, as above, we use an analytical expression for the energy build-up factor

$$B = A_1^{(\tau)}\exp(-\alpha_1^{(\tau)}\mu_\tau x) + A_2^{(\tau)}\exp(-\alpha_2^{(\tau)}\mu_\tau x). \quad (3.2)$$

Substituting (3.2) in (3.1), we get

$$K = \frac{1}{2}\gamma\left\{\int_0^{\varphi_k}\sin\varphi\sum_{i=1}^{j}\frac{\psi_{\tau i}(x)}{\mu_{\varepsilon i}}[1 - \right.$$

$$-\exp(-\mu_{\varepsilon i}b\sec\varphi)]\,d\varphi + \int_{\varphi_k}^{\varphi_0}\sin\varphi\sum_{i=1}^{j}\frac{\psi_{\tau i}(x)}{\mu_{\varepsilon i}}\times$$

$$\left.\times\left[1 - \exp\left(-\frac{\mu_{\varepsilon i}r_0}{\sin\varphi}(1 - \operatorname{tg}\varphi\operatorname{ctg}\varphi_0)\right)\right]d\varphi\right\},$$

$$\psi_{\tau i}(x) = n_i E_i\,[A_{1i}^{(\tau)}\exp(-\mu_{\tau i}'x) + A_{2i}^{(\tau)}\exp(-\mu_{\tau i}''x)],$$

$$\mu_{\tau i}' = \mu_{\tau i}(1 + \alpha_{1i}^{(\tau)}), \qquad \mu_{\tau i}'' = \mu_{\tau i}(1 + \alpha_{2i}^{(\tau)}). \quad (3.3)$$

The weight of the shield is as in (2.2). The dimensionless variables for a disc source are still valid for a cylindrical source, except that for $\varkappa$, the volume intensity γ replaces the surface intensity Γ. Then (3.3) takes the form

$$\varkappa = \int_0^{\varphi_k}\sin\varphi\sum_{i=1}^{j}\frac{\psi_{\tau i}(\zeta)}{\lambda_{\varepsilon i}}[1 -$$

$$-\exp(-\mu_{\varepsilon i}b\sec\varphi)]\,d\varphi + \int_{\varphi_k}^{\varphi_0}\sin\varphi\sum_{i=1}^{j}\frac{\psi_{\tau i}(\xi)}{\lambda_{\varepsilon i}}\times$$

$$\times\left\{1 - \exp\left[-\frac{\mu_{\varepsilon i}r_0}{\sin\varphi}(1 - \operatorname{tg}\varphi\operatorname{ctg}\varphi_0)\right]\right\}d\varphi,$$

$$\psi_{\tau i}(\zeta) = n_i\nu_i\,[A_{1i}^{(\tau)}\exp(-\lambda_{\tau i}'\zeta) +$$

$$+ A_{2i}^{(\tau)}\exp(-\lambda_{\tau i}''\zeta)], \quad \lambda_{\varepsilon i} = \mu_{\varepsilon i}/\mu. \quad (3.4)$$

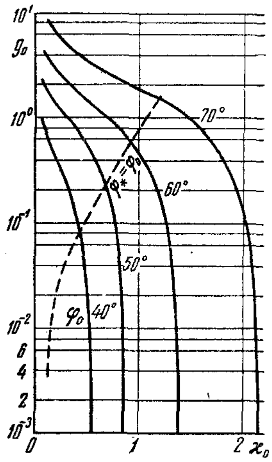

Fig. 4

In (2.4) the first equation, in correspondence with (3.4), has the form

$$\frac{d\varkappa}{d\varphi} = f(\varphi)\sin\varphi, \qquad f(\varphi) = \begin{cases} f_1(\varphi) & \text{for } 0 \leqslant \varphi \leqslant \varphi_k \\ f_2(\varphi) & \text{for } \varphi_k \leqslant \varphi \leqslant \varphi_0 \end{cases},$$

$$f_1(\varphi) = \sum_{i=1}^{j} \frac{\psi_{\tau i}(\zeta)}{\lambda_{\varepsilon i}} [1 - \exp(-\mu_{\varepsilon i} b \sec\varphi)], \qquad (3.5)$$

$$f_2(\varphi) = \sum_{i=1}^{j} \frac{\psi_{\tau i}(\zeta)}{\lambda_{\varepsilon i}} \left\{1 - \exp\left[-\frac{\mu_{\varepsilon i} r_0}{\sin\varphi}(1 - \operatorname{tg}\varphi \operatorname{ctg}\varphi_0)\right]\right\},$$

and the second equation is unchanged. The optimal control $\zeta(\varphi)$ is found from the condition of absolute maximum (see [1])

$$H^- = \operatorname{tg}\varphi\,(p_0 f_1(\varphi)\cos\varphi - \zeta\sec\varphi) \quad \text{for } 0 \leqslant \varphi \leqslant \varphi_k \quad (3.6)$$

$$H^+ = \operatorname{tg}\varphi\,(p_0 f_2(\varphi)\cos\varphi - \zeta\sec\varphi) \quad \text{for } \varphi_k \leqslant \varphi \leqslant \varphi_0\,. \quad (3.7)$$

For the first interval $(0 \leqslant \varphi \leqslant \varphi_k)$

$$p_0 \cos\varphi \sum_{i=1}^{j} \frac{\eta_{\tau i}(\zeta)}{\lambda_{\varepsilon i}} [1 - \exp(-\mu_{\varepsilon i} b \sec\varphi)] + \sec\varphi = 0\,. \qquad (3.8)$$

With the solution of (3.8) written as $\zeta_0'(\varphi)$, the optimal form of shielding is given by

$$\zeta(\varphi) = \begin{cases} \zeta_0'(\varphi) & \text{for } \Psi(\varphi) \geqslant -1/p_0 \\ 0 & \text{for } \Psi(\varphi) \leqslant -1/p_0 \end{cases} \quad (p_0 < 0)$$

$$\Psi(\varphi) = \cos^2(\varphi) \sum_{i=1}^{j} \frac{\eta_{\tau i}(0)}{\lambda_{\varepsilon i}} [1 - \exp(-\mu_{\varepsilon i} b \sec\varphi)] \qquad (3.9)$$

$$\eta_{\tau i}(\zeta) = n_i \nu_i [\lambda_{\tau i}' A_{1i}^{(\tau)} \exp(-\lambda_{\tau i}'\zeta) + \lambda_{\tau i}'' A_{2i}^{(\tau)} \exp(-\lambda_{\tau i}''\zeta)]\,.$$

The constant p_0 is obtained from (3.8) on the basis of the condition $\zeta(\varphi_*) = 0$

$$p_0 = -\frac{1}{\Psi(\varphi_*)}\,. \qquad (3.10)$$

Substituting (3.10) in (3.8), we get

$$\cos^2\varphi \sum_{i=1}^{j} \frac{\eta_{\tau i}(\zeta)}{\lambda_{\varepsilon i}} [1 - \exp(-\mu_{\varepsilon i} b \sec\varphi)] = \Psi(\varphi_*)\,. \qquad (3.11)$$

Let the solution of (3.11) be $\zeta_0(\varphi)$, then for $(0 \le \varphi \le \varphi_k)$ the optimal control $\zeta(\varphi)$ is

$$\zeta(\varphi) = \begin{cases} \zeta_0(\varphi) & \text{for } 0 \leqslant \varphi \leqslant \varphi_* \\ 0 & \text{for } \varphi_* \leqslant \varphi \leqslant \varphi_0 \end{cases} \qquad (3.12)$$

The energy flux of γ-quanta is given by *

$$\varkappa_0 = \int_0^{\varphi_*} f_1(\varphi)\sin\varphi\, d\varphi + \sum_{i=1}^{j} \frac{n_i \nu_i}{\lambda_{\varepsilon i}} \times \left\{\cos\varphi_* [1 - \Phi(\mu_{\varepsilon i} b \sec\varphi_*)] - \cos\varphi_k [1 - \Phi(\mu_{\varepsilon i} b \sec\varphi_k)]\right\} + \qquad (3.13)$$

$$\sum_{i=1}^{j} \frac{n_i \nu_i}{\lambda_{\varepsilon i}} [\cos\varphi_k - \cos\varphi_0 - X_{\varphi_k,\varphi_0}(\mu_{\varepsilon i} r_0)]\,,$$

$$X_{\varphi_k,\varphi_0}(\mu_\varepsilon r_0) = \int_{\varphi_k}^{\varphi_0} \sin\varphi \exp\left[-\frac{\mu_{\varepsilon i} r_0}{\sin\varphi}(1 - \operatorname{tg}\varphi \operatorname{ctg}\varphi_0)\right] d\varphi\,.$$

*Here $\Phi(x)$ is King's function, tabulated in [6, 7].

This integral does not exist in finite form. However, in [5] the integral

$$X(\varphi_k, \varphi_0) = \int_0^{\varphi_k} \sin\varphi \exp\left[-\frac{\mu_{\varepsilon i} r_0}{\sin\varphi}(1 - \operatorname{tg}\varphi \operatorname{ctg}\varphi_0)\right] d\varphi$$

was tabulated for a series of values of $(\mu_\varepsilon r_0)$, φ_k, φ_0.

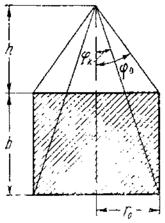

Fig. 5

The corresponding differences from this table permit the evaluation of the integral X_{φ_k,φ_0} in (3.13) since

$$X_{\varphi_k,\varphi_0} = \int_{\varphi_k}^{\varphi_0} = \int_0^{\varphi_0} - \int_0^{\varphi_k} = X(\varphi_0) - X(\varphi_k, \varphi_0)\,.$$

Equation (3.12) is valid over the range

$$\int_0^{\varphi_k} f_1(\varphi)\sin\varphi\, d\varphi + \sum_{i=1}^{j} \frac{n_i \nu_i}{\lambda_{\varepsilon i}} [\cos\varphi_k - \cos\varphi_0 - X_{\varphi_k,\varphi_0}(\mu_{\varepsilon i} r_0)] \leqslant \varkappa_0 \leqslant \sum_{i=1}^{j} \frac{n_i \nu_i}{\lambda_{\varepsilon i}} \{1 - \Phi(\mu_{\varepsilon i} b) - \cos\varphi_k [1 - \Phi(\mu_{\varepsilon i} b \sec\varphi_k)]\} +$$

$$+ \sum_{i=1}^{j} \frac{n_i \nu_i}{\lambda_{\varepsilon i}} [\cos\varphi_k - \cos\varphi_0 - X_{\varphi_k,\varphi_0}(\mu_{\varepsilon i} r_0)]\,. \qquad (3.14)$$

For the second interval $(\varphi_k \leqslant \varphi \leqslant \varphi_0)$ based on the analysis of the function H^+ (3.7), the expression

$$p_0 \cos\varphi \sum_{i=1}^{j} \frac{\eta_{\tau i}(\zeta)}{\lambda_{\varepsilon i}} \left\{1 - \exp\left[-\frac{\mu_{\varepsilon i} r_0}{\sin\varphi}(1 - \operatorname{tg}\varphi \operatorname{ctg}\varphi_0)\right]\right\} + \sec\varphi = 0 \qquad (3.15)$$

is obtained.

Denoting the solution of (3.15) as $\zeta_2'(\varphi)$, the optimal form of shielding for $(\varphi_k \leqslant \varphi \leqslant \varphi_0)$ is

$$\zeta(\varphi) = \begin{cases} \zeta_2'(\varphi) & \text{for } T(\varphi) \geqslant -1/p_0 \\ 0 & \text{for } T(\varphi) \leqslant -1/p_0 \end{cases} \quad (p_0 < 0)$$

$$T(\varphi) = \cos^2\varphi \sum_{i=1}^{j} \frac{\eta_{\tau i}(0)}{\lambda_{\varepsilon i}} \left\{1 - \exp\left[-\frac{\mu_{\varepsilon i} r_0}{\sin\varphi}(1 - \operatorname{tg}\varphi \operatorname{ctg}\varphi_0)\right]\right\} \qquad (3.16)$$

$$0 \leqslant T(\varphi) \leqslant \cos^2\varphi_k \sum_{i=1}^{j} \frac{\eta_{\tau i}(0)}{\lambda_{\varepsilon i}} [1 - \exp(-\mu_{\varepsilon i} b \sec\varphi_k)] \quad \text{for } \varphi_k \leqslant \varphi \leqslant \varphi_0\,.$$

Since at $\varphi = \varphi_0$, $T(\varphi)$ vanishes, the intersection $\varphi = \varphi_*$ always lies in the interval $[0, \varphi]$ and the peripheral regions always remain unshielded.

As above, p_0 is determined from the condition $\zeta(\varphi_*) = 0$. On this basis, Eq. (3.15) gives

$$p_0 = -\frac{1}{T(\varphi_*)}. \quad (3.17)$$

Substituting from (3.17) in (3.8) and (3.15), we get

$$\cos^2\varphi \sum_{i=1}^{j} \frac{\eta_{\gamma i}(\xi)}{\lambda_{\varepsilon i}} [1 - \exp(-\mu_{\varepsilon i} b \sec\varphi)] = T(\varphi_*), \quad (3.18)$$

$$\cos^2\varphi \sum_{i=1}^{j} \frac{\eta_{\gamma i}(\xi)}{\lambda_{\varepsilon i}} \Big\{1 - \exp\Big[-\frac{\mu_{\varepsilon i} r_0}{\sin\varphi}(1 - \operatorname{tg}\varphi \operatorname{ctg}\varphi_0)\Big]\Big\} = T(\varphi_*). \quad (3.19)$$

The optimal form of shielding for $(0 \le \varphi \le \varphi_0)$ is

$$\zeta(\varphi) = \begin{cases} \zeta_1(\varphi) & \text{for } 0 \leqslant \varphi \leqslant \varphi_k \\ \zeta_2(\varphi) & \text{for } \varphi_k \leqslant \varphi \leqslant \varphi_* \\ 0 & \text{for } \varphi_* \leqslant \varphi \leqslant \varphi_0, \end{cases} \quad (3.20)$$

where $\zeta_1(\varphi)$ and $\zeta_2(\varphi)$ are the solutions of (3.18) and (3.19).

Equation (3.20) is valid in the γ-quanta energy flux interval

$$0 < \varkappa_0 \leqslant \int_0^{\varphi_k} f_1(\varphi) \sin\varphi d\varphi + \sum_{i=1}^{j} \frac{n_i v_i}{\lambda_{\varepsilon i}} [\cos\varphi_k - \cos\varphi_0 - X_{\varphi_k, \varphi_0}(\mu_{\varepsilon i} r_0)]. \quad (3.21)$$

The weight of the optimal shield is determined by integrating the second equation in (2.4). $\zeta(\varphi)$ is substituted from (3.12) on the interval (3.14), and from (3.20) on the interval (3.21).

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